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On the nonexistence of iterative roots

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Abstract

For the germ of a holomorphic mapping $F:(U,0)\to(\mathbb{C},0)$ of the form $F(z)=\varrho z+\cdots$, where ϱ is a primitive root of unity of order $d\geqslant 2$, criteria for the existence of a continuous iterative root of given order and the topological linearizability of F are given.

The following conditions are equivalent: (1) $F^d=\mathrm{Id}$; (2) the germ of the mapping $F(z)$ is topologically conjugate to the germ of the mapping ϱz; (3) the germ of the mapping F has a continuous iterative root of order d^k for every $k\geqslant 1$.

If $F^d\neq\mathrm{Id}$, then for a given positive integer N the germ of the mapping F has a continuous iterative root of order N iff $d\cdot\gcd(N,\mathrm{ind}(F^d,0)-1)$ divides $\mathrm{ind}(F^d,0)-1$. © 1997 Elsevier Science B.V.

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0. Introduction

The problem of embedding a given mapping $F:X\to X$ in a one-parameter group (semigroup) of mappings $f_\lambda:X\to X$ is best examined for the case of the group of all real numbers $\lambda\in\mathbb{R}$. Indeed, many positive embedding results are obtained for the group of all complex numbers $\lambda\in\mathbb{C}$, and the nonexistence of embedding is usually obtained on the level of the group of all rational numbers $\lambda\in\mathbb{Q}$ (see the equivalence of statements (i) and (iv) in 2°, Section 3. Because in the general case the embedding problem cannot have a solution with a satisfactory formulation, this problem is usually considered in various classes of mappings. We will consider germs of local mappings (maybe, formal) $F:(U,\bar{0})\to(\mathbb{R}^m,\bar{0})$, that is, mappings F defined on some neighborhood $U\subseteq\mathbb{R}^m$ of the point $\bar{0}$; they are given by a formal (maybe, nonconvergent) series with zero

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free coefficient. Mappings coinciding on a certain neighborhood of the fixed point $\bar{0}$ are identified. The best known general results can be divided into the following three groups.

(1) The Poincaré theorem says that *every formal (complex) mapping F with linear part without resonances is formally linearizable*, [1, p. 175], [13, p. 39], [104], [94], [88, Theorem 1], [27], and therefore, *embeddable in a formal $\mathbb{C}$-flow*.

Similarly, *every C^∞-mapping F with linear part without resonances is C^∞-linearizable* [113] and therefore, *embeddable in an $\mathbb{R}$-flow of C^∞-mappings*.

S. Sternberg [113], F. Takens [119], G.R. Belitskii [12], [13, pp. 148, 152], G.R. Sell [106] and many other authors [98,115,20,16] proved similar theorems on C^k-linearization.

The problem of topological linearization of a variable mapping is solved by the following Grobman–Hartman theorem [13, p. 148]:

Let $A:\mathbb{R}^m \to \mathbb{R}^m$ be a nondegenerate linear operator. The germ

$$F(x) = Ax + g(x), \quad g(\bar{0}) = \bar{0}, \ Dg(\bar{0}) = \bar{0},$$

of any C^1-diffeomorphism is topologically (that is, by a homeomorphism $H:(\mathbb{R}^m,\bar{0}) \to (\mathbb{R}^m,\bar{0})$) conjugate to the linear diffeomorphism Ax when and only when the spectrum of the operator A does not lie on the unit circle.

Therefore, a hyperbolic mapping F is embeddable into an $\mathbb{R}$-flow of continuous mappings. However, the question of the possibility to linearize a particular nonhyperbolic mapping is open and undoubtedly, very complicated.

The analytical linearization is examined by A. Poincaré, C.L. Siegel [109], [1, pp. 176, 194–199], [47], A.D. Brjuno [18,19], J.-C. Yoccoz [126], [77, p. 282] and many other mathematicians [114,123,124,45,127]. The one-dimensional complex case was first treated by E. Schröder [100] and G. Koenigs [54]. Thus, even if the eigenvalues of the A operator do not have resonances (and the analytical F mapping can therefore be formally linearized), the analytical-linearization problem rests on the number-theoretic properties of the eigenvalues. For example, the linearizability of the simplest one-dimensional (complex one-dimensional) holomorphic mapping $F(z) = \exp(2\pi i\alpha)z + \cdots$, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ depends on the number-theoretic properties of the number α (G.A. Pfeiffer [79], H. Cremer [28–30], C.L. Siegel [109], A.D. Brjuno [18,19], J.-C. Yoccoz [126], R. Pérez-Marco [75–78]).

(2) *Every (maybe, infinite) square complex upper-triangular matrix with a fixed nonzero element $a_{ii} = \varrho \neq 0$ on the diagonal is embeddable into an analytical $\mathbb{C}$-flow* (L. Reich [87], L. Reich and J. Schwaiger [94]). This theorem has deep applications to examining the existence of formal iterative roots of given order with a given multiplicator (see 1° in Section 3).

(3) *If a formal mapping $F:(U,0) \to (\mathbb{C},0)$ is not the identity and is tangent to the identity mapping at zero, that is, has the form $F(z) = z + a_m z^m + \cdots$, where $a_m \neq 0$, $m \geqslant 2$, then it is embeddable into a formal $\mathbb{C}$-flow* (I.N. Baker [5,8]). I.N. Baker [8], G. Szekeres [117], and P. Erdős and E. Jabotinsky [43] considered the formal series $f_\lambda(z) = z + \lambda a_m z^m + \cdots$, where the terms starting with the third one are uniquely determined by the formal equation $f_\lambda \circ F = F \circ f_\lambda$.

Here we give some conditions showing that certain classes of mappings cannot be embedded into a flow. For every flow and every positive integer N, we have $F = f_1 = (f_{1/N})^N$, therefore, the existence of an iterative root of order N is necessary for an affirmative answer to the embedding question.

It is the problem of the existence of a continuous iterative root of a given order N that is our concern in this work. We consider the problem of finding the conditions on the germ of a mapping F ensuring the possibility of the existence of a germ of a continuous mapping $G:(U,\bar{0}) \to (\mathbb{R}^m,\bar{0})$ such that for a given positive integer N, the equality

$$G^N = F$$

holds, where G^N stands for the N-fold iteration of the G mapping.

In finding the necessary conditions, we will use the topological notions of the *multiplicity* $\deg(F,\bar{0})$ *of a germ of the mapping* F *at zero* for an isolated preimage $\bar{0}$ of zero and the *index* $\mathrm{ind}(F,\bar{0})$ *of a fixed point* $\bar{0}$ *of a germ of the mapping* F for an isolated fixed point $\bar{0}$ of the F mapping. Our methods allow us to formulate the necessary conditions for the existence of a continuous iterative root of a given order N, which are nontrivial if the linear component of the F mapping is degenerate or the eigenvalues of the linear component (the differential of F at zero) include primitive roots of unity of order $\geqslant 2$.

The existence of an smooth iterative root of order N of a linear operator $\Lambda:\mathbb{R}^m \to \mathbb{R}^m$ is equivalent to the existence of the root of order N of the operator matrix (which can be verified by a simple procedure). Therefore, the found obstacles to the existence of an iterative root imply obstacles to the topological linearization of the given mapping.

Because evaluating of the multiplicity of zero and index of a fixed point requires thoroughly considering the spectrum of a linear approximation and certain monomials of high degree in the (formal) decomposition of F, we will restrict our consideration by complex one-dimensional mappings. Thus the author [15] demonstrated the use of the *index* (ind) in examining the existence of a formal (smooth) iterative root of *given order with a given multiplicator*. As a multiplicator is not defined for a continuous (nonsmooth) mapping, our conditions correspond to a smooth mapping with the weakest condition on the mapping F.

Theorem 0.1. *For a germ of a holomorphic mapping* $F:(U,0) \to (\mathbb{C},0)$ *of the form* $F(z) = a_m z^m + \cdots$, *where* $a_m \neq 0$, $m \geqslant 2$, *and a positive integer* N, *the following conditions are equivalent:*

(1) *There exists a positive integer* k *such that* $m = k^N$.
(2) *The mapping* F *has a formal iterative root of order* N.
(3) *The mapping* F *has a holomorphic iterative root of order* N.
(4) *The mapping* F *has a continuous iterative root of order* N.

Theorem 0.2. *For a germ of a holomorphic mapping* $F:(U,0) \to (\mathbb{C},0)$ *of the form* $F(z) = \varrho z + \cdots$, *where* ϱ *is a primitive root of unity of order* $d \geqslant 2$ *such that* $F^d \neq \mathrm{Id}$, *and a positive integer* N, *the following conditions are equivalent:*

(1) *The number* $d \cdot \gcd(N, \mathrm{ind}(F^d,0) - 1)$ *divides the number* $\mathrm{ind}(F^d,0) - 1$.
(2) *The mapping* F *has a formal iterative root of order* N.

(3) *The mapping F has a continuous iterative root of order N.*

(4) *The mapping F has exactly* $\gcd(N, \operatorname{ind}(F^d, 0) - 1)$ *formal iterative roots of order N. These roots are formally nonconjugate.*

(5) *The mapping F has at least* $\gcd(N, \operatorname{ind}(F^d, 0) - 1)$ *continuous iterative roots of order N. These roots are topologically nonconjugate.*

Theorem 0.3. *For a germ of a holomorphic mapping $F:(U,0) \to (\mathbb{C},0)$ of the form $F(z) = \varrho z + \cdots$, where ϱ is a primitive root of unity of order $d \geqslant 2$, the following conditions are equivalent:*

(1) $F^d = \mathrm{Id}$.

(2) *The mapping $F(z)$ is formally conjugate to the mapping ϱz.*

(3) *The mapping $F(z)$ is holomorphically conjugate to the mapping ϱz.*

(4) *For every N, there exist $n \geqslant N$ and a formal mapping G of the form $G(z) = \varepsilon_n z + \cdots$, where ε_n is a primitive root of unity of order n, such that $G^n = F^d$.*

(5) *For every N, there exist $n \geqslant N$ and a C^1-mapping $G:(U,\bar{0}) \to (\mathbb{R}^2,\bar{0})$ of the form $G(z) = \varepsilon_n z + g(z,\bar{z})$, $g(0,0) = \bar{0}$, $Dg(0,0) = \bar{0}$, where ε_n is a primitive root of unity of order n, such that $G^n = F^d$.*

(6) *The mapping F has a continuous iterative root of order d^k for every $k \geqslant 1$.*

1. Super-attracting fixed point

Let $F:(U,\bar{0}) \to (\mathbb{R}^m,\bar{0})$ be an admissible mapping, that is, a continuous mapping such that $F^{-1}(\bar{0}) \cap U = \bar{0}$. Then for some $\varepsilon_0 > 0$, the mapping $G: B_{\varepsilon_0}(\bar{0}) \setminus \bar{0} \to S^{m-1}$ given by the formula $G(z) = F(z)/\|F(z)\|$ is well defined and the degree of the mapping $G|_{S_\varepsilon^{m-1}(\bar{0})}: S_\varepsilon^{m-1}(\bar{0}) \to S^{m-1}$ does not depend on the choice of $\varepsilon \leqslant \varepsilon_0$. It is called the *degree* of the mapping F at the point $\bar{0}$ and denoted as $\deg(F,\bar{0})$ [26]. We need the following:

Proposition 1.1. *If $f:(U,\bar{0}) \to (\mathbb{R}^m,\bar{0})$ and $g:(V,\bar{0}) \to (\mathbb{R}^m,\bar{0})$ are admissible mappings, then the composition $g \circ f:(W,\bar{0}) \to (\mathbb{R}^m,\bar{0})$ is also an admissible mapping and* $\deg(g \circ f,\bar{0}) = \deg(g,\bar{0}) \cdot \deg(f,\bar{0})$.

Proposition 1.2. *Let $F:(U,0) \to (\mathbb{C},0)$ be a holomorphic mapping such that $F(z) = a_m z^m + \cdots$, where $a_m \neq 0$, $m \geqslant 1$. Then F is admissible and* $\deg(F,0) = m$.

Proof of Theorem 0.1. The equivalence of (1) and (2) is asserted in the Schwaiger theorem [102]. The implication (3) $\Rightarrow$ (4) is evident. We only have to prove the implications (1) $\Rightarrow$ (3) and (4) $\Rightarrow$ (1).

(1) $\Rightarrow$ (3). According to the Bötkher theorem [17], [53, pp. 154–156], [57, Théoréme 1 ($r = 1$)], [11, Theorem 6.10.1], [77, p. 275], the mapping F is analytically conjugated to the mapping $\widetilde{F}(z) = z^m$. The mapping $\widetilde{F}(z) = z^m$ has a holomorphic iterative root of order N — $\widetilde{G}(z) = z^k$. Therefore, the conjugated mapping F also has a holomorphic iterative root of order N.

(4) $\Rightarrow$ (1). Let $G:(U,0) \to (\mathbb{C},0)$ be a germ of a continuous mapping such that $G^N = F$. As the mapping F is admissible, the mapping G is also admissible. Therefore, the number $k = \deg(G,0)$ is defined. Then by Propositions 1.1 and 1.2, we have

$$m = \deg(F,0) = \deg(G^N,0) = (\deg(G,0))^N = k^N. \qquad \square$$

Remark 1.3. J. Schwaiger [102] showed that if condition (1) holds, then the F mapping has exactly $1 + k + \cdots + k^{N-1} = (m-1)/(k-1)$ formal iterative roots of order N. If in proving the implication (1) $\Rightarrow$ (3), we take the mapping $\widetilde{G}_\lambda(z) = \lambda z^k$ for λ satisfying the condition $\lambda^{(m-1)/(k-1)} = 1$, then we obtain exactly $(m-1)/(k-1)$ holomorphic iterative roots of order N. According to the Schwaiger theorem the germ mapping F has no additional formal root and the more so, additional holomorphic root. According to the Bötkher theorem all this holomorphic iterative roots are holomorphically conjugate.

2. Rationally indifferent fixed point

For a germ of a continuous mapping $F:(U,\bar{0}) \to (\mathbb{R}^m,\bar{0})$ such that the only fixed point of the mapping F is zero, the *index* is

$$\operatorname{ind}(F,\bar{0}) = \deg(\mathrm{Id} - F,\bar{0}).$$

We need the following property of an index proved by H. Steinlein [112], P.P. Zabreiko and M.A. Krasnosel'skii [55], and A. Dold [31].

Proposition 2.1 (see [14]). *Let $F:(U,\bar{0}) \to (\mathbb{R}^m,\bar{0})$ be a germ of a continuous mapping such that for every positive integer n, the point $\bar{0}$ is isolated in the set of all fixed points of the mapping F^n. Then there exists a sequence of integers $\{A_d\}_{d=1}^{\infty}$ such that*

$$\operatorname{ind}(F^n,\bar{0}) = \sum_{d\mid n} d\cdot A_d \quad \textit{for every } n \geqslant 1.$$

Lemma 2.2. *Let a holomorphic mapping $F:(U,0) \to (\mathbb{C},0)$ have the form $F(z) = \varrho z + \cdots$, where $\varrho = \exp(2\pi i a/d)$ with $d = p^\alpha$ and $\gcd(a,d) = 1$ for some prime number $p \geqslant 2$ and natural $\alpha \geqslant 1$. If the mapping F has a continuous iterative root of order $N = p^\beta$, then either*

$$F^d = \mathrm{Id}, \quad \textit{or} \quad \operatorname{ind}(F^d,0) \equiv 1 \pmod{p^{\alpha+\beta}}.$$

Proof. Let $F^d(z) \not\equiv z$. Then $F^d(z) = z + a_m z^m + \cdots$ for some $m \geqslant 2$ and $a_m \neq 0$. The point 0 is an isolated fixed point of the map F^d and therefore [108], [14, Proposition 1, Theorem 1], there exists a natural number $A \geqslant 1$ such that

$$\operatorname{ind}(F^n,0) = \begin{cases} 1, & d \nmid n, \\ m = 1 + Ad, & d \mid n. \end{cases}$$

Let $G:(U,0) \to (\mathbb{C},0)$ be a continuous map such that $G^N = F$. Then the point 0 is an isolated fixed point of G^n for every positive integer n, and by Proposition 2.1, there exists a sequence of integers $\{B_k\}_{k=1}^{\infty}$ such that $\operatorname{ind}(G^n,0)=\sum_{k|n} k\cdot B_k$. Hence, the equality $G^N=F$ implies

$$\sum_{k|p^{\beta+i}} k\cdot B_k = 1 \quad \text{for } i=0,\dots,\alpha-1 \quad \text{and} \quad \sum_{k|p^{\alpha+\beta}} k\cdot B_k = 1+p^{\alpha}\cdot A.$$

Hence, $B_k=0$ for $k=p^{\beta+1},\dots,p^{\alpha+\beta-1}$ and $p^{\alpha+\beta}B_{p^{\alpha+\beta}}=p^{\alpha}A$, that is, $A=p^{\beta}B_{p^{\alpha+\beta}}$. We therefore proved that $\operatorname{ind}(F^d,0)=1+dA=1+p^{\alpha+\beta}B_{p^{\alpha+\beta}}$. □

Corollary 2.3. *Let a holomorphic mapping $F:(U,0)\to(\mathbb{C},0)$ have the form $F(z)=\varrho z+\cdots$, where ϱ is a primitive root of unity of order $d=p_1^{\alpha_1}\cdots p_m^{\alpha_m}\geqslant 2$. If the mapping F has a continuous iterative root of order $N=p_1^{\beta_1}\cdots p_m^{\beta_m}N'$, then either*

$$F^d=\mathrm{Id},\quad \text{or}\quad \operatorname{ind}(F^d,0)-1\equiv 0 \ \left(\operatorname{mod} p_1^{\alpha_1+\beta_1}\cdots p_m^{\alpha_m+\beta_m}\right).$$

Proof. Let $F^d(z)\not\equiv z$. The mapping $F_i=F^{p_1^{\alpha_1}\cdots p_{i-1}^{\alpha_{i-1}}p_{i+1}^{\alpha_{i+1}}\cdots p_m^{\alpha_m}}$ satisfies the conditions of Lemma 2.2 and has a continuous iterative root of order $p_i^{\beta_i}$. Therefore,

$$\operatorname{ind}(F^d,0)=\operatorname{ind}\left(F_i^{p_i^{\alpha_i}},0\right)\equiv 1 \ \left(\operatorname{mod} p_i^{\alpha_i+\beta_i}\right).$$

As the numbers $\{p_i\}_{i=1}^m$ are mutually prime, the required congruence is equivalent to the system of obtained congruences. □

Remark 2.4. The divisibility of the number $\operatorname{ind}(F^d,0)-1$ by some divisors of N' can be derived only from the consideration of the multiplicator of the iterative root G, which is impossible for nonsmooth mappings.

Lemma 2.5. *Let ϱ be a primitive root of unity of order $d\geqslant 2$, and let the decomposition of the positive integer d in primes be $d=p_1^{\alpha_1}\cdots p_m^{\alpha_m}$. Let N be a positive integer such that $N=p_1^{\beta_1}\dots p_m^{\beta_m}N'$ and $\gcd(N',d)=1$. Then for a positive integer B, the following conditions are equivalent:*

(1) *The congruence $B\equiv 0 \ (\operatorname{mod} p_1^{\alpha_1+\beta_1}\cdots p_m^{\alpha_m+\beta_m})$ holds.*
(2) *The number $d\cdot\gcd(N,B)$ divides B*
(3) *There exists a number λ such that $\lambda^N=\varrho$ and $\lambda^B=1$.*
(4) *There are exactly $\gcd(N,B)$ different numbers λ such that $\lambda^N=\varrho$ and $\lambda^B=1$.*

Proof. Let the number ϱ have an additional representation

$$\varrho=\exp(2\pi i a/d),\quad \gcd(a,d)=1.$$

(1) ⇒ (2). There exists a positive integer B' such that

$$B=p_1^{\alpha_1+\beta_1}\cdots p_m^{\alpha_m+\beta_m}\cdot B'.$$

The number $d\cdot\gcd(N,B)=p_1^{\alpha_1}\cdots p_m^{\alpha_m}p_1^{\beta_1}\cdots p_m^{\beta_m}\cdot\gcd(N',p_1^{\alpha_1}\cdots p_m^{\alpha_m}\cdot B')=p_1^{\alpha_1+\beta_1}\cdots p_m^{\alpha_m+\beta_m}\cdot\gcd(N',B')$ divides the number $B=p_1^{\alpha_1+\beta_1}\cdots p_m^{\alpha_m+\beta_m}\cdot B'$.

(2) ⇒ (1). Let $B=p_1^{\gamma_1}\cdots p_m^{\gamma_m}\cdot B'$, where $\gcd(d,B')=1$. As the number $d\cdot\gcd(N,B)$ divides the number B, the number $p_i^{\alpha_i+\min(\beta_i,\gamma_i)}$ also divides B, and therefore, $\alpha_i+\min(\beta_i,\gamma_i)\leqslant\gamma_i$. The last inequality implies that $\min(\beta_i,\gamma_i)=\beta_i$ and $\alpha_i+\beta_i\leqslant\gamma_i$.

(1) $\Rightarrow$ (3). As d and N' are mutually prime, there exist positive numbers k and l such that $lN' - kd = a$. Let us now consider the primitive root

$$\lambda = \exp\left(2\pi\mathrm{i}\frac{l}{p_1^{\alpha_1+\beta_1}\cdots p_m^{\alpha_m+\beta_m}}\right) = \exp(2\pi\mathrm{i}(a+kd)/dN)$$

of unity of order $p_1^{\alpha_1+\beta_1}\cdots p_m^{\alpha_m+\beta_m}$. Then $\lambda^N = \exp(2\pi\mathrm{i}(a+kd)/d) = \varrho$ and $\lambda^B = 1$.

(3) $\Rightarrow$ (1). As $\lambda^N = \varrho$, there exist an integer k such that $\lambda = \exp(2\pi\mathrm{i}(a+kd)/dN)$. From $\lambda^B = 1$ we derive that $((a+kd)/dN)B = l$ is an integer number. As $a+kd$ is mutually prime with the number $p_1\ldots p_m$, the congruence $B \equiv 0 \ (\operatorname{mod} p_1^{\alpha_1}\cdots p_m^{\alpha_m}\cdot p_1^{\beta_1}\cdots p_m^{\beta_m})$ holds.

(3) $\Rightarrow$ (4). If for some integer k the number $\lambda_{(k)} = \exp(2\pi\mathrm{i}(a+kd)/dN)$ is such that $\lambda_{(k)}^N = \varrho$ and $\lambda_{(k)}^B = 1$, then the number

$$\lambda_{(k+N/\gcd(N,B))} = \exp\bigl(2\pi\mathrm{i}\bigl(a+(k+N/\gcd(N,B))d\bigr)/dN\bigr)$$

also satisfies the condition (4). Conversely, let $\lambda_{(k)}$ and $\lambda_{(k')}$ be two numbers satisfying the condition (4). Then the numbers $(a+kd)B/dN$ and $(a+k'd)B/dN$ are integer. Therefore the number $(k-k')dB/dN = (k-k')B/N$ is integer, that is $(k-k') \equiv 0 \ (\operatorname{mod} N/\gcd(N,B))$.

The implication (4) $\Rightarrow$ (3) is evident. $\square$

Proof of Theorem 0.2. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (4). These equivalences are reformulations of the theorems proved by B. Muckenhoupt [73, Theorem 6] and J. Schwaiger [102, Theorem 3b].

(3) $\Rightarrow$ (1). According to Lemma 2.5, this implication is exactly Corollary 2.3.

(1) $\Rightarrow$ (5). By the Camacho theorem [22, Theorem 1], [33, p. 46–47], [11, Theorem 6.10.6], the mapping $F(z)$ is topologically conjugate to the mapping

$$\widetilde{F}(z) = \varrho z\bigl(1 + z^{\operatorname{ind}(F^d,0)-1}\bigr).$$

Let the number λ be such that $\lambda^N = \varrho$ and

$$\lambda^{\operatorname{ind}(F^d,0)-1} = 1.$$

According to Lemma 2.5 there are exactly $\gcd(N, \operatorname{ind}(F^d,0)-1)$ such numbers different from each other. The mapping

$$\widetilde{G}_\lambda(z) = \lambda z\bigl(1 + z^{\operatorname{ind}(F^d,0)-1}\bigr)$$

is holomorphic. By [103, Lemma 1], [15, Lemma 2], its Nth iteration has the form $\widetilde{G}_\lambda^N(z) = \lambda^N z(1+\cdots) = \varrho z(1+\cdots)$, and by [103, Lemma 1], [15, Lemmas 2–3], its (Nd)th iteration has the form

$$\widetilde{G}_\lambda^{Nd}(z) = \lambda^{Nd} z\bigl(1 + Ndz^{\operatorname{ind}(F^d,0)-1} + \cdots\bigr) = z\bigl(1 + Ndz^{\operatorname{ind}(F^d,0)-1} + \cdots\bigr).$$

By the Camacho theorem [22], the mappings F and $\widetilde{G}_\lambda^N$ are topologically conjugate. This means that there exist a local homeomorphism $H:(U,0) \to (\mathbb{R}^2,0)$ such that $F = H^{-1}\circ\widetilde{G}_\lambda^N\circ H$. Then the germ $G_\lambda = H^{-1}\circ\widetilde{G}_\lambda\circ H$ is a continuous iterative root of the germ F of order N. So we constructed exactly $\gcd(N, \operatorname{ind}(F^d, 0) - 1)$ different continuous iterative roots of germ F of order N. As the germs $\widetilde{G}_\lambda$ have different multiplicators, they are topologically nonconjugate by the Naïshul' theorem [74,44] and therefore, all constructed roots G_λ are topologically nonconjugate. For the applicability of the Naïshul' theorem, it should also be noted that different numbers λ satisfying the condition (4) of Lemma 2.5 are not complex conjugate.

The implication (5) $\Rightarrow$ (3) is evident. □

Remark 2.6. In the work [102], J. Schwaiger proves the equivalence of conditions (1) and (2) using a modification of Lemma 2.5.

Remark 2.7. In the works [49], [65], [121, Theorem 2.1.20], the problem of the existence of an iterative root of given order N for a given bijection $F : X \to X$ of an arbitrary set X onto itself was solved completely. However, the obtained iterative roots are usually discontinuous. Note that the obstacles to the existence of a discontinuous root of order N found by S. Łojasiewicz are an analog of Proposition 2.1 (in the discrete case, the index of an open set is equal to the number of fixed points in this set). Thus, the Łojasiewicz theorem implies that the mapping $F_m : \mathbb{Z} \to \mathbb{Z}$ given by $F_m(n) = n + m$ has an iterative root of order N iff $m \equiv 0 \pmod N$. Now let

$$X = \{0\} \cup \bigcup_{i=-\infty}^{+\infty} \left[2^{2i}, 2^{2i+1}\right] \quad \text{and} \quad F(x) = 4^{-1}x.$$

The mapping F is a contraction and therefore, it has discontinuous iterative roots of all orders [121, Proposition 2.1.22], but it does not have a continuous iterative root of any order $N \geqslant 2$. Indeed, let a continuous mapping $G : X \to X$ be such that $G^N = F$. Because G is continuous, $G([2^{2i}, 2^{2i+1}])$ lies in one connected component of the space X, therefore, the induced mapping $\widehat{G} : \{*\} \cup \mathbb{Z} \to \{*\} \cup \mathbb{Z}$ is well defined. The equality $G^N = F$ implies that $\widehat{G}^N = \widehat{F}$, but as was mentioned, the mapping $F_1 = \widehat{F}$ does not have iterative roots of order $N \geqslant 2$. Note that the mapping $4^{-1}\mathrm{Id}$ is linear on the real line and therefore, it can be embedded into the continuous $\mathbb{R}$-flow of linear mappings $4^{-r}\mathrm{Id}$.

It is shown in the works [95,105,25] that the mapping $F(z) = az^2 + bz + c$, $a \neq 0$, of the plane does not have any (including discontinuous) iterative roots. In the works [49,46,2,3,111,96,97,128,129], the problem of the existence of an iterative root of a given order of an arbitrary self-mapping of an arbitrary set is considered.

The problem of the existence of continuous iterative roots of a continuous mapping of the real line into itself is discussed in work [56] and in even more detail, in monograph [58].

I.N. Baker [4], [6, p. 152], [7, p. 284] proved the equality of the derivatives at fixed points belonging to the same orbit of an iterative root which has periodic points of all orders except, possibly, one [7], [11, Theorem 6.2.2] to be an obstacle to the existence of an entire root of a given order.

Proof of Theorem 0.3. The implications (3) $\Rightarrow$ (2) $\Rightarrow$ (1), (2) $\Rightarrow$ (4), (3) $\Rightarrow$ (5), and (3) $\Rightarrow$ (6) are evident. We only have to prove the implications (1) $\Rightarrow$ (3), (4) $\Rightarrow$ (1), (5) $\Rightarrow$ (1), and (6) $\Rightarrow$ (1).

(1) $\Rightarrow$ (3). This implication is contained in [110, pp. 159–161], [73, Lemma 4], [77, p. 276].

(4) $\Rightarrow$ (1). This implication in a little different form was communicated to the author by L. Reich. (Cf. [88, Theorems 7–9], [91], [92].)

Let us assume that $F^d(z) \not\equiv z$. Then $F^d(z) = z + a_m z^m + \cdots$ for some $m \geqslant 2$ and $a_m \neq 0$. There exist $n \geqslant m$ and a formal mapping G of the form $G(z) = \varepsilon_n z + \cdots$, where ε_n is a primitive root of unity of order n, such that $G^n = F^d$. The mapping G has the semicanonical form $\widetilde{G}(z) = \varepsilon_n z(1 + a_{kn} z^{kn} + \cdots)$ for some positive integer k. Hence, the mapping G^n has the semicanonical form $\widetilde{G}^n(z) = z(1 + n a_{kn} z^{kn} + \cdots)$. From the uniqueness of the degree of the second nonvanishing term in the semicanonical form, we obtain $kn + 1 = m$. This contradicts the choice of $n \geqslant m$.

(5) $\Rightarrow$ (1). Let us assume that $F^d(z) \not\equiv z$. Then $F^d(z) = z + a_m z^m + \cdots$ for some $m \geqslant 2$ and $a_m \neq 0$. There exist $n \geqslant m$ and a C^1-mapping $G:(U,\bar{0}) \dashrightarrow (\mathbb{R}^2,\bar{0})$ of the form $G(z) = \varepsilon_n z + g(z,\bar{z})$, $g(0,0) = \bar{0}$, $Dg(0,0) = \bar{0}$, where ε_n is a primitive root of unity of order n, such that $G^n = F^d$. By Proposition 2.1 and the Shub–Sullivan theorem [108], [14, Theorem 1], there exists an integer number A such that $1 + nA = \operatorname{ind}(G^n,\bar{0}) = \operatorname{ind}(F^d,\bar{0}) = m$. This contradicts the choice of $n \geqslant m$.

(6) $\Rightarrow$ (1). Let us assume that $F^d(z) \not\equiv z$. Then $F^d(z) = z + a_m z^m + \cdots$ for some $m \geqslant 2$ and $a_m \neq 0$. According to Theorem 0.2 and Lemma 2.5, the number d^{k+1} divides the number $m - 1 = \operatorname{ind}(F^d,\bar{0}) - 1$ for every integer $k \geqslant 0$. This condition shows that $m = \infty$, that is, $F^d(z) \equiv z$. $\square$

Remark 2.8. If the numbers $\deg(F,0)$ and $\operatorname{ind}(F^d,0)$ are finite, then they are determined by a finite jet of the germ of the mapping F. Therefore, for Theorems 0.1 and 0.2, an analog of the Reich "stability theorem" [89, Theorem 2], [15, Theorem 2] can be formulated. Theorem 0.3 also admits of introducing "stable" versions of the corresponding properties.

3. The mapping F is tangent to the identity mapping

As we mentioned in the introduction, any germ of a mapping tangent to the identity mapping is embeddable into a $\mathbb{C}$-flow of formal mappings. I.N. Baker [5,8] and P. Erdős and E. Jabotinsky [43] examined the structure of the set of all complex $\lambda \in \mathbb{C}$ such that the mapping f_λ has positive convergence radius.

Baker's theorem [8, Theorems 2, 10 and 11]. *If $F(z)$ is any formal series of the form*

$$F(z) = z + \sum_{n=m}^{\infty} a_n z^n, \quad a_m \neq 0,\ m \geqslant 2, \tag{TId}$$

then for every formal series $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ *permutable with* F ($F \circ g = g \circ F$), *we have* (1) $b_2 = \ldots = b_{m-1} = 0$, (2) b_m *is arbitrary and* (3) *for an arbitrary value of* λ, *there is exactly one* $g(z)$ *with* $b_m = \lambda a_m$. *This* $g(z)$ *is denoted as* $f_\lambda(z)$ *and has the form*

$$f_\lambda(z) = z + \lambda a_m z^m + \sum_{n=m+1}^{\infty} b_n(\lambda) z^n, \tag{Fam}$$

where $b_n(\lambda)$ *is a polynomial in* λ *of degree at most* $(n - m + 1)$.

If S *is the set of* λ *corresponding to the convergent members of the family* (Fam), *it has one of the forms*

(i) *the point* $\lambda = 0$,

(ii) $\{n\lambda_0\}$, $n = 0, \pm 1, \pm 2, \ldots$ *with* $\lambda_0 \neq 0$ (*a linear set*),

(iii) $\{n\lambda_0 + m\lambda_1\}$, $n = 0, \pm 1, \ldots$, $m = 0, \pm 1, \ldots$, $\lambda_0 \neq 0$, $\lambda_1 \neq 0$, λ_1/λ_0 *is not real* (*a plane lattice*), *or*

(iv) *the whole plane.*

For the mapping

$$F(z) = e^z - 1 = z + \tfrac{1}{2}z^2 + \tfrac{1}{6}z^6 + \cdots,$$

the mapping $f_\lambda(z)$ *from the family* (Fam) *have positive radius of convergence iff the complex number* λ *is one of* $0, \pm 1, \pm 2, \ldots$.

The family

$$f_\lambda(z) = \frac{z}{1 - \lambda z} = \sum_{n=0}^{\infty} \lambda^n z^{n+1} \tag{Fl}$$

of mappings forms an analytic $\mathbb{C}$*-flow of holomorphic mappings.*

I.N. Baker [8, pp. 285–286] also constructed a formal series $F(z)$ for which the series $f_\lambda(z)$ converges in a circle of a positive radius only for $\lambda = 0$, and J. Écalle [36,39] (and S.M. Voronin for $m = 2$ [125]) showed that case (iii), that is, the case of the two-dimensional lattice, cannot occur. In the works [8–10,118,63,86,34], it is shown that for many special classes of maps, the situation (ii) occurs. I.N. Baker [8, p. 288] and P. Erdős and E. Jabotinsky [43] associate a series $L(z)$ ($I(z)$) with a mapping $F(z)$ of the form (TId), which is a formal solution to the functional equation $LF(z) = F'(z) \cdot L(z)$ with its first nonzero term equal to the second nonzero term of $F(z)$. The series $L(z)$ can also be represented as

$$L(z) = \left.\frac{\partial f_\lambda(z)}{\partial \lambda}\right|_{\lambda=0}.$$

I.N. Baker [8], P. Erdős and E. Jabotinsky [43,50–52] showed that the first k coefficients of a series determine the first k coefficients of its associated series, and vice versa. In addition, the associated series has positive radius of convergence if and only if all series f_λ have positive radius of convergence. E. Jabotinsky examined the properties of the coefficients of the series $L(z)$ [52].

Many authors obtained the formal canonical normal form of a mapping of the form (TId) [73, Theorem 1], [120, Theorem 2], [125, Theorem 1.1], [81, p. 113], [11, Theorem 6.10.7].

Every mapping F of the form (TId) *is formally conjugate to a mapping* $\widetilde{F}(z) = z + z^m + cz^{2m-1}$, *and this canonical normal form is unique.*

As a corollary to the cited results, B. Muckenhoupt [73] proved that the mappings $F(z) = e^z - 1$ and $H(z) = z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \cdots$, whose associated function is $L(z) = \frac{1}{2}z^2 - \frac{1}{12}z^3$, are formally conjugate and holomorphically nonconjugate. The problems of holomorphic conjugacy of germs of holomorphic mappings, inclusion of a germ of holomorphic mapping into a $\mathbb{C}$-flow of germs of holomorphic mappings, and the existence of a holomorphic iterative root of given order are solved by J. Écalle [35,37–42,66] and S.M. Voronin ($m = 2$) [125,107,66,69].

The mentioned holomorphic $\mathbb{C}$-flow (Fl) has the property that every two mappings of this flow (except $f_0(z) \equiv z$) are linearly (and hence, holomorphically) conjugate. Indeed, $\lambda^{-1}(F(\lambda z)) = f_\lambda(z)$. This example appears to be an exclusive one.

Theorem 3.1. *For a formal mapping F of the form* (TId), *two iterates f_λ and f_μ, $\lambda \neq 0$, $\mu \neq 0$, are formally conjugate iff either $\lambda = \mu$, or the mapping $F(z)$ is formally conjugate to the mapping $\widetilde{F}(z) = z + z^m + (m/2)z^{2m-1}$.*

Proof. Let the formal canonical normal form of the mapping F be $\widetilde{F}(z) = z + z^m + cz^{2m-1}$. Then the mapping $f_\lambda(z)$ has the form

$$f_\lambda(z) = z + \lambda z^m + \left[\lambda c + \frac{\lambda(\lambda-1)}{2}m\right]z^{2m-1} + \cdots.$$

The coefficients of the formal series $f_\lambda(z)$ can be evaluated by inductively considering monomials in the left- and right-hand sides of the equation $f_\lambda \circ F = F \circ f_\lambda$.

The functional identity

$$(z + az^m + cz^{2m-1}) \circ \lambda z = \lambda z \circ (z + \lambda^{m-1}az^m + \lambda^{2m-2}cz^{2m-1})$$

and the uniqueness of the canonical normal form show that $z + az^m + cz^{2m-1}$ and $z + bz^m + dz^{2m-1}$ are formally conjugate iff $c/a^2 = d/b^2$. Hence, f_λ and f_μ are formally conjugate iff

$$\left[\lambda c + \frac{\lambda(\lambda-1)}{2}m\right]/\lambda^2 = \left[\mu c + \frac{\mu(\mu-1)}{2}m\right]/\mu^2,$$

that is,

$$\begin{aligned}\mu c + \mu(\lambda-1)\frac{m}{2} = \lambda c + \lambda(\mu-1)\frac{m}{2} &\Leftrightarrow (\mu-\lambda)c = \frac{m}{2}(\mu-\lambda)c\\ &\Leftrightarrow \left(c - \frac{m}{2}\right)(\mu-\lambda) = 0\\ &\Leftrightarrow c = \frac{m}{2}, \text{ or } \mu = \lambda. \qquad \Box\end{aligned}$$

Iterations proved to be always topologically conjugate. Mappings of form (TId) and moreover, mappings as in Theorem 0.2 are topologically classified by C. Camacho [22].

We used his classification in the proof of Theorem 0.2. The case $\varrho = 1$, $m = 2$ was also considered by A.A. Sherbakov [107].

Theorem 3.2. *A germ of a holomorphic mapping F of the form* (TId) *is embeddable in a continuous $\mathbb{C}$-flow f_λ of germs of continuous mappings. All complex iterates f_λ, $\lambda \neq 0$, in this flow are topologically conjugate.*

Proof. By the Camacho theorem, all mappings of form (TId) with fixed $m \geqslant 2$ are topologically conjugate, and the number m is a topological invariant. (This is implied by the equality $m = \operatorname{ind}(F, 0)$.) Then, the given germ of the mapping F is topologically conjugate to the mapping $H(z) = z + z^m + (m/2)z^{2m-1} + \cdots$, whose associated series is $L(z) = z^m$. By the Baker–Erdős–Jabotinsky theorem, the mapping H has an analytic (because the dependence of coefficients of iterate h_λ on λ is analytic) $\mathbb{C}$-flow of germs of holomorphic mappings. The topological conjunction of the $\mathbb{C}$-flow h_λ gives a continuous $\mathbb{C}$-flow of germs of continuous mappings f_λ for which $f_1 = F$.

The mappings h_λ, $\lambda \neq 0$, are of type (iv) (from the Baker theorem), therefore, the Baker–Erdős–Jabotinsky theorem implies that associated series of these mappings have a positive radius of convergence. According to our Theorem 3.1, these series are formally conjugate, and by the Muckenhoupt theorem [73, Theorem 2], they are also holomorphically conjugate. Note that the topological conjugacy of the mappings h_λ, $\lambda \neq 0$, is an immediate corollary to the Camacho theorem. Therefore, mappings f_λ, $\lambda \neq 0$, are also topologically conjugate. □

Remark 3.3. The notion of index and multiplicity is also useful in considering multidimensional mappings. The conditions $F^d \equiv \mathrm{Id}$ and $F \not\equiv 0$ should be then replaced by the conditions "0 is not an isolated fixed point of the mapping F^d" and "0 is an isolated preimage of zero", which are readily verified.

General results on the existence of iterative roots for multidimensional mappings are apparently exhausted by the following four situations.

1°. L. Reich and J. Schwaiger obtained a very strong *criterion for the existence of a formal iterative root of order N with a given multiplicator* in terms of semicanonical normal forms [94,88,89].

2°. Let F be an automorphism of the complex formal power series ring $\mathbb{C}[[z_1, \ldots, z_m]]$. Joint efforts of D.N. Lewis [64], S. Sternberg [114], L. Reich and J. Schwaiger [87,93,94,88], W. Bucher [21], G.H. Mehring [70,71], and C. Praagman [80–84] made it possible to state that the following conditions are equivalent:

(i) *F is embeddable in an analytic $\mathbb{C}$-flow.*

(ii) *F is embeddable in a continuous $\mathbb{R}$-flow.*

(iii) *F is embeddable in a continuous $\mathbb{Q}$-flow.*

(iv) *F has roots of all orders.*

(v) *F is conjugate to an automorphism in smooth normal form.*

(vi) *F is conjugate to an automorphism in normal form which has roots of all orders in normal form.*

(vii) *F is an exponential of a derivation of* $\mathbb{C}[[z_1, \ldots, z_m]]$, *i.e., F has a logarithm.*

3°. *For every formal mapping $F:(U,\bar{0}) \to (\mathbb{C}^m, \bar{0})$ with a nondegenerate differential at $\bar{0}$, there exists a positive integer d such that for F^d, each statement* (i)–(vii) *is fulfilled* [64,24,101].

4°. L. Reich and A.R. Kräuter [91,92] showed that *for a formal mapping F, the presence of special iterative roots implies the embeddability of F into an analytic $\mathbb{C}$-flow.*

Remark 3.4. K.T. Chen [24] showed that for hyperbolic real C^∞-germs, there is an analog of 3°. He also constructed a C^∞-mapping $F:(U,0) \to (\mathbb{R}^3, 0)$ for any positive integer d which has hyperbolic fixed point and is such that the mapping F^k satisfies the condition (ii) from 2° for $k = d$ and does not satisfy the condition (ii) from 2° for $k = 1, \ldots, d-1$. Considering nonhyperbolic fixed points makes it possible to reduce the dimensionality of the "implementation" space. (For a plane hyperbolic C^∞-germ, we always have $d = 1$ or $d = 2$ [24].)

Theorem 3.5. *For any integer $d \geqslant 2$, there exists a holomorphic germ $F:(U,0) \to (\mathbb{C}, 0)$ such that for a given positive integer n, the following conditions are equivalent:*

(1) $n \equiv 0 \pmod d$.

(2) *The germ F^n can be embedded into an analytic $\mathbb{C}$-flow of germs of holomorphic mappings f_λ.*

(3) *The germ F^n has a continuous iterative root of order d.*

(4) *The germ F^n has a formal iterative root of order d.*

Proof. Consider the germ of the holomorphic mapping

$$H(z) = z\left[1 + z^d + \frac{d+1}{2} z^{2d} + \cdots\right],$$

whose associated series is $L(z) = z^{d+1}$. By the Baker–Erdős–Jabotinsky theorem, the mapping $H(z)$ has an analytic $\mathbb{C}$-flow of germs of holomorphic mappings. As was proved by B. Muckenhoupt [73, p. 166], the mapping $H(z)$ has the form

$$H(z) = z\left[1 + z^d + \sum_{n=2}^{\infty} b_n z^{nd}\right].$$

It is easy to see from the form of the decomposition of the germ of H that the germs $H(z)$ and $R_d = \varrho z$, where ϱ is a primitive root of unity of order d, are commutative. Consider the map

$$F(z) = R_d \circ H(z) = \varrho z\left[1 + z^d + \sum_{n=2}^{\infty} b_n z^{nd}\right].$$

As the germs R_d and H are commutative, we have $F^n = R_d^n \circ H^n$ for all n.

(α) Let $n = dn'$. Then $F^n = R_d^{dn'} \circ H^n = H^n$, and the map $H^n = h_n$ is embedded into an analytic $\mathbb{C}$-flow of germs of holomorphic mappings. (It suffices to put $(f)_\lambda = h_{n\lambda}$.)

(β) Let $\gcd(n,d) < d$ and $\tilde{d} = d/\gcd(n,d) \geqslant 2$. Then

$$F^n = R_d^n \circ H^n(z) = \varrho^n z\left[1 + nz^d + \cdots\right] = \varepsilon_{\tilde{d}} z\left[1 + nz^d + \cdots\right],$$

where $\varepsilon_{\tilde{d}}$ is a primitive root of unity of order $\tilde{d} \geqslant 2$. Because the number $\tilde{d}\gcd(d,d) = \tilde{d}d$ does not divide d, Theorem 0.2 implies that the germ F^n does not have an iterative root (neither formal, nor continuous) of order d. □

Remark 3.6. In (3) and (4), the order d of an iterative root can be replaced by an order d' such that the number $\gcd(d', d)$ does not divide $\gcd(n, d)$.

Remark 3.7. Gy. Targonski [122] and J. Schwaiger [103] examined the existence of so-called "phantom" roots.

Remark 3.8. In the work [90], the problem of *evaluation* of *all* formal iterative roots of a formal mapping $F(z) = \varrho z + \cdots$, where ϱ is a root of unity, is solved.

4. Irrationally indifferent fixed point

Now, let a germ of a holomorphic mapping $F:(U,0) \to (\mathbb{C},0)$ have the form

$$F(z) = \varrho z + \cdots, \quad \text{where } \varrho = \exp(2\pi i\alpha) \text{ with } \alpha \in \mathbb{R}\setminus\mathbb{Q}. \tag{Irr}$$

In the case under consideration, resonances are absent, therefore, *any mapping as in* (Irr) *is formally linearizable*. For the first time, the existence of holomorphically nonlinearizable germs was proved by G.A. Pfeiffer [79]. His proof showed that in constructing a nonlinearizable germ, the approximability of the multiplicator ϱ by roots of unity is very important.

For an irrational number $\alpha \in \mathbb{R}\setminus\mathbb{Q}$, consider the following "arithmetical" objects [77,78]. The norm of α is

$$\|\alpha\| = \inf_{p\in\mathbb{Z}} |\alpha - p|.$$

By induction, a sequence $\{\alpha_n\}_{n=0}^{\infty}$ is defined:

$$\alpha_0 = \|\alpha\| \quad \text{and} \quad \alpha_n = \left\|\alpha_{n-1}^{-1}\right\| \quad \text{for } n \geqslant 1;$$
$$\beta_{-1} = 1 \quad \text{and} \quad \beta_n = \prod_{j=0}^{n} \alpha_j \quad \text{for } n \geqslant 0.$$

From the sequences constructed, the following two numbers (possibly, infinite) are determined, which play an important role in the dynamics of the germ F:

$$\Phi(\alpha) = \frac{1}{2\pi}\sum_{n\geqslant 0} \beta_{n-1}\log\alpha_n^{-1},$$
$$\Psi(\alpha) = \frac{1}{2\pi}\sum_{n\geqslant 0} \beta_{n-1}\log\log \mathrm{e}\,\alpha_n^{-1}.$$

From the representation of the number α in the form of an infinite chain fraction, two integer sequences $\{q_n\}_{n=0}^{\infty}$ and $\{\tilde{q}_n\}_{n=0}^{\infty}$ can be determined, which can also be specified by the inductive formulas [78, p. 573]

$$q_{n+1} = \min\left\{q \geqslant 1\colon\ \|q\alpha\| < \tfrac{1}{2}\|q_n\alpha\|\right\} \quad \text{and}$$
$$\tilde{q}_{n+1} = \min\left\{q \geqslant 1\colon\ \|q\alpha\| < \|\tilde{q}_n\alpha\|\right\}.$$

The following propositions are contained in R. Pérez-Marco's works in the most complete and explicit forms.

Proposition 4.1 (see [126, p. 55], [77, p. 278], [78, p. 568]). *For the number*

$$\varrho = \exp(2\pi \mathrm{i}\alpha) \quad \text{with } \alpha \in \mathbb{R} \setminus \mathbb{Q},$$

the following conditions are equivalent:

$$\liminf_{n\to\infty} \sqrt[n]{|\varrho^n - 1|} = 0, \tag{Cr}$$

$$\sup \frac{\log q_{n+1}}{q_n} = +\infty. \tag{Cr$_2$}$$

Proposition 4.2 (see [77, p. 278], [78, pp. 568, 613]). *For the number $\varrho = \exp(2\pi\mathrm{i}\alpha)$ with $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, the following conditions are equivalent:*

$$\liminf_{n\to\infty} \sqrt[d^n]{|\varrho^n - 1|} = 0, \tag{d}$$

$$\sup \frac{\log q_{n+1}}{d^{q_n}} = +\infty. \tag{d$_2$}$$

Proposition 4.3 (see [32, p. 152], [126, p. 55]). *For $\varrho = \exp(2\pi\mathrm{i}\alpha)$ with $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, the following conditions are equivalent:*

there exist $C, \tau > 0$ such that
for all rational p/q, we have $|\alpha - p/q| \geqslant C/q^{\tau}$, (Dioph)

there exist $C, \tau > 0$ such that
for all $n \geqslant 0$, we have $q_{n+1} \leqslant C q_n^{\tau}$, (Dioph$_2$)

$$\sup \frac{\log q_{n+1}}{\log q_n} < +\infty. \tag{Dioph$_3$}$$

Proposition 4.4 (see [126, p. 56], [75, p. 534], [77, p. 306], [78, pp. 574, 635]). *For*

$$\varrho = \exp(2\pi\mathrm{i}\alpha) \quad \text{with } \alpha \in \mathbb{R} \setminus \mathbb{Q},$$

the following conditions are equivalent:

$$\sum_{n\geqslant 0}\frac{\log q_{n+1}}{q_n} < +\infty, \tag{Br}$$

$$\sum_{n\geqslant 0}\frac{\log \tilde{q}_{n+1}}{\tilde{q}_n} < +\infty, \tag{Br$_2$}$$

$$\Phi(\alpha) < +\infty, \tag{Yoc}$$

Proposition 4.5 (see [75, p. 534], [78, pp. 574, 635]). *For the number*

$$\varrho = \exp(2\pi \mathrm{i}\alpha) \quad \textit{with } \alpha \in \mathbb{R}\setminus\mathbb{Q}.$$

the following conditions are equivale .t:

$$\sum_{n\geqslant 0}\frac{\log\log q_{n+1}}{q_n} < +\infty, \tag{Per}$$

$$\sum_{n\geqslant 0}\frac{\log\log \tilde{q}_{n+1}}{\tilde{q}_n} < +\infty, \tag{Per$_2$}$$

$$\Psi(\alpha) < +\infty \tag{Per$_\Psi$}$$

Proposition 4.6 (see [77, p. 279], [78, p. 612]). *For* $\varrho = \exp(2\pi\mathrm{i}\alpha)$ *with* $\alpha \in \mathbb{R}\setminus\mathbb{Q}$, *the following conditions are equivalent:*

$$\liminf_{n\to\infty} \sqrt[d^n]{|\varrho^n - 1|} = 0 \quad \textit{for every positive } d, \tag{∞}$$

$$\sup \frac{\log\log q_{n+1}}{q_n} = +\infty. \tag{∞_2}$$

In works published in 1928, 1935, and 1938, H. Cremer [28–30] suggested nice and simple approaches to constructing holomorphically nonlinearizable germs. Thus, if the multiplicator ϱ satisfies the condition (Cr), then there exist a nonlinearizable germ of the entire mapping. But if the multiplicator ϱ does satisfy the condition (d), then every polynomial germ $F(z) = \varrho z + \cdots + a_d z^d$, $a_d \neq 0$, and every rational mapping

$$F(z) = \frac{\varrho z + a_2 z^2 + \cdots + a_r z^r}{1 + b_1 z + \cdots + b_s z^s} \tag{Rat$_\varrho$}$$

with $r < s = d$, $b_d \neq 0$, has the property that the fixed point 0 is the limit of periodic points and therefore, none of such mappings is linearizable.

C.L. Siegel proved in 1942 [109] that if the rotation angle α of the multiplicator ϱ is Diophantian, that is, it satisfies the condition (Dioph), then every germ of the form (Irr) is linearizable. A.D. Brjuno [18,19] strengthened the Siegel theorem by proving the linearizability of each germ of the form (Irr) for the case when the number α satisfies the condition (Br). J.-C. Yoccoz estimated the radius of convergence of a linearizing formal mapping in the case of the fulfillment of the Brjuno condition with the help of the number $\Phi(\alpha)$ [126, p. 56], [77, pp. 281, 283]. J.-C. Yoccoz also showed that for

$\Phi(\alpha) = \infty$, the mapping $\varrho z + z^2$ is not linearizable [126, p. 56], [75, p. 533]. In 1988, he showed that the fixed point is then the limit of periodic points, and in 1989 [77, p. 283], [78, p. 605 (Théoréme IV.1.1)] that the fixed point is even the limit of periodic orbits. Using Yoccoz's method, R. Pérez-Marco showed that for any nonlinearizable germ of the form (Irr) whose multiplicator satisfies the condition (Per), the fixed point is the limit of periodic orbits [75, Théorème 2], [77, p. 298], [78, Théorème 3]. For $\Psi(\alpha) = \infty$, he *constructed a germ as in* (Irr) *defined and one-to-one on a unit disk such that if the iterations of some point are inside the unit disk, then they converge to the fixed point* [75, Théoréme 1], [77, p. 298], [78, Théoréme 2]. R. Pérez-Marco also used Yoccoz's method to show that if the condition (∞) is fulfilled, then for any polynomial nonlinear germ the fixed point 0 is the limit of periodic *orbits* [78, Proposition IV.3.1]. He also obtained theorems on nonlinearizability of structurally stable polynomial germs under a weaker condition imposed on the multiplicator ϱ [78, Théorème 4].

Note that the problems of topological and holomorphic linearization of a germ of the form (Irr) are equivalent. Various forms of the following assertion can be found in many works.

Theorem 4.7. *For a germ of holomorphic mapping as in* (Irr), *the following conditions are equivalent:*

(1) *The germ F is holomorphically linearizable.*

(2) *The germ F is linearizable by a local homeomorphism.*

(3) *The mapping F has an open invariant simply connected neighborhood of zero whose boundary contains more than one point.*

Proof. The implications $(1) \Rightarrow (2) \Rightarrow (3)$ are evident. We only have to prove the implication $(3) \Rightarrow (1)$.

$(3) \Rightarrow (1)$. Let U be an open invariant simply connected neighborhood of zero with more than one point on the boundary. By the Riemann mapping theorem [67, Theorem III.1.2 (v. III, pp. 8, 12)], there exists a conformal mapping $H : U \to K$ of the neighborhood U onto a unit disk K with the center at the origin such that $H(0) = 0$. Then $\widetilde{F} = H \circ F \circ H^{-1}$ is a mapping of K into itself such that $\widetilde{F}(0) = 0$ and $\widetilde{F}'(0) = \exp(2\pi i\alpha)$.

By the Schwarz lemma [67, Theorem I.17.8 (v. I, p. 381)], we have $\widetilde{F}(z) = \exp(2\pi i\alpha)z$. $\square$

Note that in many works [85], [68, p. 63], [116, p. 736], the property of "the existence of periodic points tending to zero" is mentioned, which presents obstacles for topological linearizability. However, because the orbit of a periodic point close to the fixed point can include points outside the circle where the conjugation is defined, if there is no invariant neighborhood (i.e., the germ of the mapping F is nonlinearizable), the difference in the behavior of the periods of periodic points converging to the fixed point is not an obstacle to the conjugacy of two nonlinearizable germs. This shows the necessity of obtaining theorems on the existence of periodic *orbits* converging to the fixed point.

Lemma 4.8. *Let $F:(U,0)\to(\mathbb{C},0)$ be a germ of a rational mapping as in* (Rat_ϱ) *and $F_n(z)$ be the nth iteration of F. Then F_n has the form*

$$F_n(z)=\frac{\varrho^n z+\cdots+a_r^{1+r+r^2+\cdots+r^{n-1}}z^{r^n}}{1+\cdots+b_s^{1+(r-s)+\cdots+(r-s)^{n-1}}a_r^{r+\cdots+r^{n-1}-(r-s)-\cdots-(r-s)^{n-1}}z^{r^n-(r-s)^n}},$$

when $r>s$; (1)

$$F_n(z)=\frac{\varrho^n z+\cdots+\varrho^{n-1}a_r b_s^{s^{n-1}-1}z^{s^n+r-s}}{1+\cdots+b_s^{s^{n-1}}z^{s^n}}\quad \textit{when } r<s. \tag{2}$$

The validity of formulas (1) and (2) is verified by induction on the number n.

Theorem 4.9. *If the number $\varrho=\exp(2\pi i\alpha)$, where $\alpha\in\mathbb{R}\setminus\mathbb{Q}$, satisfies the condition*

$$\liminf_{n\to\infty}\sqrt[d^n]{|\varrho^n-1|}\,(1+\delta)^n=0 \quad \textit{for some } \delta>0, \tag{d, δ}$$

then for each germ of a rational mapping as in (Rat_ϱ) *satisfying one of the conditions:*

(1) $r<s\leqslant d$,

(2) $s+1<r\leqslant d$,

(3) $s+1=r\leqslant d$ *and* $|a_r|\neq|b_s|$,

each neighborhood V of the fixed point 0 contains a periodic point z such that $F^k(z)\in V$ for all k, i.e., the point 0 is then the limit of periodic orbits.

Proof. Let us show that any preset neighborhood V of the fixed point 0 contains points that are fixed with respect to the iteration F_n. Bringing the equation $F_n(z)=z$ to a common denominator and dividing by z (we seek fixed points different from zero) yields the equations

$$\varrho^n+\cdots+A_{(n)}z^{s^n+r-s-1}=1+\cdots+b_s^{s^{n-1}}z^{s^n}\quad \text{in the case (1)},$$

$$\varrho^n+\cdots+a_r^{1+r+\cdots+r^{n-1}}z^{r^n-1}=1+\cdots+B_{(n)}z^{r^n-(r-s)^n}\quad \text{in the case (2)},$$

$$\varrho^n+\cdots+a_r^{1+r+\cdots+r^{n-1}}z^{r^n-1}=1+\cdots+a_r^{1+r+\cdots+r^{n-1}-n}b_s^n z^{r^n-1}$$

in the case (3).

Moving all monomials into one side of the equation and collecting similar terms gives the equations

$$\varrho^n-1+\cdots-b_s^{s^{n-1}}z^{s^n}=0\quad \text{in the case (1)}$$

$$\varrho^n-1+\cdots+a_r^{1+r+\cdots+r^{n-1}}z^{r^n-1}=0\quad \text{in the case (2)}$$

$$\varrho^n-1+\cdots+a_r^{1+r+\cdots+r^{n-1}}\bigl(1-(b_s/a_r)^n\bigr)z^{r^n-1}=0\quad \text{in the case (3)}.$$

By the Viète–Bezout theorem on the product of roots of a polynomial, our equations have roots with moduli not exceeding

$$\left|\frac{\varrho^n-1}{b_s^{s^{n-1}}}\right|^{1/s^n}=\frac{|\varrho^n-1|^{1/s^n}}{|b_s|^{1/s}}\leqslant\frac{|\varrho^n-1|^{1/d^n}}{|b_s|^{1/s}}\quad \text{in the case (1)},$$

$$\left|\frac{\varrho^n-1}{a_r^{(r^n-1)/(r-1)}}\right|^{1/(r^n-1)} = \frac{|\varrho^n-1|^{1/(r^n-1)}}{|a_r|^{1/(r-1)}} \leqslant \frac{|\varrho^n-1|^{1/d^n}}{|a_r|^{1/(r-1)}} \quad \text{in the case (2)},$$

$$\left|\frac{\varrho^n-1}{a_r^{(r^n-1)/(r-1)}(1-(b_s/a_r)^n)}\right|^{1/(r^n-1)} = \frac{|\varrho^n-1|^{1/(r^n-1)}}{|a_r|^{1/(r-1)}|1-(b_s/a_r)^n|^{1/(r^n-1)}}$$

$$\leqslant \frac{|\varrho^n-1|^{1/d^n}}{|a_r|^{1/(r-1)}|1-(b_s/a_r)^n|^{1/(r^n-1)}} \quad \text{in the case (3)}.$$

As $F'(0)=\exp(2\pi i\alpha)$, there exists a neighborhood $B_\varepsilon(0)$ of 0 such that $|F'(z)| \leqslant 1+\delta$ for all points $z \in B_\varepsilon(0)$, where δ is the positive number from the condition (d,δ). It can always be assumed that $B_\varepsilon(0) \subset V$. The condition (d,δ) implies that there exists a number n such that

$$|\varrho^n-1|^{1/d^n}(1+\delta)^n \leqslant A\varepsilon, \tag{ε}$$

where $A=|b_s|^{1/s}$ in the case (1), $A=|a_r|^{1/(r-1)}$ in the case (2), and

$$A=|a_r|^{1/(r-1)}|1-(b_s/a_r)^n|^{1/(r^n-1)}$$

in the case (3). In the case (3), we have $|a_r| \neq |b_s|$, therefore, for sufficiently large numbers n, the inequality $|1-(b_s/a_r)^n|^{1/(r^n-1)} \geqslant 2^{-1}$ holds, and it can be assumed that in all cases the coefficient A does not depend on n and the inequality (ε) does have solutions.

Therefore, in all three cases, there exist a point z_0 such that $F^n(z_0)=z_0$ and $|z_0| \leqslant \varepsilon/(1+\delta)^n$, and for $z_1=F(z_0), \ldots, z_{k+1}=F(z_k)$ with $k \leqslant n-1$, we will have $|F(z_k)-F(0)| \leqslant |F'(\tilde{z}_k)| \cdot |z_k-0|$. Hence, $|z_{k+1}| \leqslant (1+\delta)\cdot\varepsilon/(1+\delta)^{n-k} \leqslant \varepsilon$, which implies that the entire orbit of the periodic point z_0 (remember, $z_n=z_0$) lies inside the ε-neighborhood of the fixed point 0. □

Corollary 4.10. *If the number $\varrho=\exp(2\pi i\alpha)$, where $\alpha \in \mathbb{R}\setminus\mathbb{Q}$, satisfies the condition (∞), then for any germ of a rational mapping of the form (Rat_ϱ) with $r \neq s$, $s+1$, or with $r=s+1$ and $|a_r| \neq |b_s|$, any neighborhood V of the fixed point 0 contains a periodic point z such that $F^k(z) \in V$ for all k, i.e., the point 0 is then the limit of periodic orbits.*

Proof. Indeed, the condition (∞) implies the condition (d,δ). □

Remark 4.11. Corollary 4.10 is a generalization of similar Pérez-Marco's assertion for polynomial germs formulated above. Our proof of Theorem 4.9 follows proof by H. Cremer, who showed the existence of periodic *points* in any neighborhood of the fixed point under a weaker requirement to the number ϱ (the multiplier $(1+\delta)^n$ is not required). Thereby, H. Cremer constructed first particular examples of nonlinearizable mappings (even at the level of mapping of the form $\varrho z+z^2$).

Corollary 4.12. *If the number* $\varrho = \exp(2\pi \mathrm{i}\alpha)$, *where* $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, *satisfies the conditions* (d, δ) *and* $\Psi(\alpha) = \infty$, *then the Pérez-Marco germ* $G(z) = \varrho z + \cdots$ *without periodic orbits* (*except* 0) *is not topologically conjugate to any rational mapping as in* (Rat_σ) *satisfying one of the following conditions:*

(1) $r < s \leqslant d$,

(2) $s + 1 < r \leqslant d$,

(3) $s + 1 = r \leqslant d$ *and* $|a_r| \neq |b_s|$.

Proof. Let $G(z) = \varrho z + \cdots$ be a Pérez-Marco germ and $F(z)$ be a rational map of the form (Rat_σ) with $r \neq s$, $s + 1$, or $r = s + 1$ and $|a_r| \neq |b_s|$. If the germs G and F are topologically conjugate, then by the Naĭshul' theorem, their multiplicators are equal: $\varrho = \sigma$. Then by the Theorem 4.9, some periodic orbits of the mapping F converge to the fixed point 0. But since this property is invariant with respect to topological conjugacy, we obtain a contradiction with the properties of the Pérez-Marco germ. □

Corollary 4.13. *If the number* $\varrho = \exp(2\pi \mathrm{i}\alpha)$, *where* $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, *satisfies the condition* (∞), *then the Pérez-Marco germ* $G(z) = \varrho z + \cdots$ *without periodic orbits* (*except* 0) *is not topologically conjugate to any rational mapping of the form* (Rat_σ) *with* $r \neq s, (s + 1)$, *or with* $r = s + 1$ *and* $|a_r| \neq |b_s|$.

Remark 4.14. In Theorem 4.9 and Corollaries 4.10, 4.12 and 4.13, the restrictions on the rational function F can be weakened. For example, if $r = s + 1$, $|a_r| = |b_s|$, $b_s/a_r = \exp(2\pi \mathrm{i}\beta)$, and β is Diophantian, all these assertions remain valid. Indeed, we than have

$$B = \liminf |1 - (b_s/a_r)^n|^{1/(r^n - 1)} > 0.$$

According to the Cremer theorem, the condition (d) (with $d = r$) would then contradict the Siegel linearizability theorem, and in the proof of Theorem 4.9, it would suffice to put $A = B/2$. Corollary 4.13 implies that the Pérez-Marco germ is not topologically conjugate to any polynomial germ and is topologically finitely nondetermined. It would be interesting to learn whether or not *any Pérez-Marco germ can be topologically* (*or holomorphically*) *conjugate to any rational germ.* For $r = s$ or $r = s+1$, Theorem 4.9 can be valid only under certain additional conditions. For $r = s = 1$, the germ $\varrho z/(1 + bz)$ is linearizable, and for $r = s + 1 = 1$, the germ ϱz is linear. The answers to the following questions are unknown to the author. *Is it true that the mentioned cases exhaust linearizable rational germs of irrationally indifferent fixed point? Is Corollary* 4.10 *valid for a nonlinearizable rational germ?*

J.-C. Yoccoz constructed an uncountable family of classes of holomorphic conjugacy of germs of the form (Irr) containing no integer function for any multiplicator ϱ with $\Phi(\alpha) = \infty$. He also examined a multidimensional analog of the Siegel theorem [77, p. 282].

Note that by the Naĭshul' theorem [74,44], *for the map as in* (Irr) (*with an arbitrary real* α), *the number* $\exp(2\pi \mathrm{i}\alpha)$ *is a topological invariant.* If F is an irrational rotation, the topological invariance of an "indifferent" multiplicator is well known. In the multidimensional case, it is proved by N.H. Kuiper and J.W. Robbin [60]. The works [61,62,48,23,59] describe topological invariants for linear $\mathbb{C}$-flows.

A nonlinearizable germ of an irrationally indifferent fixeu point can never be embedded into a $\mathbb{C}$-flow. Therefore, in considering the problem of embedding of the germ F, it makes more sense to consider the center $Z(F)$ of this germ.

Proposition 4.15. *If two formal germs $G(z) = r\exp(2\pi i\alpha)z$ and $F(z) = a_1z + \cdots$ are commutative, then either $F(z) = a_1z$, or $r = 1$ and $\alpha \in \mathbb{Q}$.*

This is proved by comparing the coefficients of the powers of z in the formal series $G \circ F(z)$ and $F \circ G(z)$.

Remark 4.16. Assertions similar to Proposition 4.15 and the Naĭshul' theorem are not valid for continuous mappings of the plane even for a quadratic mapping whose linear part is a rotation by an angle α.

Proposition 4.17. *For any mappings $r:[0,\infty) \to [0,\infty)$ and $\varphi:[0,\infty) \to [0,2\pi)$, the mapping $G(z) = r(|z|)e^{i\varphi(|z|)}z$ is commutative with the mapping $R_\alpha(z) = e^{2\pi i\alpha}z$. And vice versa, if $\alpha \in \mathbb{R}\setminus\mathbb{Q}$, then any germ commutative with R_α has this form.*

Proof. Indeed, let $F \circ R_\alpha = R_\alpha \circ F$. Then

$$F \circ R_\alpha^n(z) = R_\alpha^n \circ F(z) \quad \text{for all } n \geqslant 1. \tag{com}$$

Let the complex number z satisfy the condition $z = |z|$. If $z = 0$, then $F(0) = R_\alpha(F(0))$, i.e., we then have $F(0) = 0$. If $z = |z| > 0$, then the sequence $\{R_\alpha^n(z)\}_{n\geqslant 0}$ forms an everywhere dense subset of a circle of radius $|z|$, and the sequence $\{R_\alpha^n(F(z))\}_{n\geqslant 0}$ forms an everywhere dense subset of a circle of radius $|F(z)| \geqslant 0$. Since the mapping F is continuous, the image of a circle of radius $|z|$ lies on a circle of radius $|F(z)|$. Put $r(|z|) = |F(z)|/|z|$. Let $\varphi(|z|)$ denote the argument of the complex number $F(z)$. Then the equality (com) implies

$$F\big(e^{2\pi in\alpha}|z|\big) = e^{2\pi in\alpha}e^{i\varphi(|z|)}|F(z)| = r(|z|)e^{i\varphi(|z|)}\big(e^{2\pi in\alpha}|z|\big) = G\big(e^{2\pi in\alpha}|z|\big).$$

The continuity of the F mapping and the density of the sequence $\{e^{2\pi in\alpha}|z|\}_{n\geqslant 0}$ on a circle of radius $|z|$ imply the coincidence of the mappings F and G on the entire circle. $\square$

Proposition 4.18 (see [76, p. 461], [77, p. 300]). *If $F(z)$ is a nonlinearizable germ of the form* (Irr) *and $G(z) = a_1z + \cdots$ is in $Z(F)$, then*

(1) $|a_1| = 1$,

(2) *a_1 is non-Diophantian,*

(3) *if a_1 is a root of unity of order d, then $G^d = \mathrm{Id}$,*

(4) *the germ $G(z)$ is uniquely determined by the multiplicator a_1.*

Proof. (1) If $|a_1| \neq 0,\ 1$, then there exists a conformal mapping $H:(U,0)\to(\mathbb{C},0)$ such that $\widetilde{G}(z)=H^{-1}\circ G\circ H(z)=a_1z$. By Proposition 4.15, the germ $\widetilde{F}(z)=H^{-1}\circ F\circ H(z)$ has the form ϱz, that is, the germ F is linearizable.

If $a_1=0$, then $\widetilde{G}(z)=z^m$. The equality $\widetilde{F}\circ\widetilde{G}=\widetilde{G}\circ\widetilde{F}$ (that is, $\varrho z^m+\cdots=\varrho^m z^m+\cdots$) implies $\varrho=\varrho^m$. This means that $\varrho=0$, or $\varrho^{m-1}=1$.

(2) If a_1 is Diophantian, then by the Siegel theorem, we have $\widetilde{G}(z)=a_1z$, and by Proposition 4.15, the germ $\widetilde{F}(z)$ is linear.

(3) If a_1 is a root of unity of order d, then $G^d(z)=z+\cdots$. If $G^d(z)=z+a_mz^m+\cdots$, $a_m\neq 0$, $m\geqslant 2$, then the comparison of the coefficients of z^{m+1} in the series $F\circ G^d(z)$ and $G^d\circ F(z)$ yields the equality $\varrho^{m-1}=1$ [8, Theorem 1].

(4) The comparison of the coefficients of powers of z in $F\circ G(z)$ and $G\circ F(z)$ makes it possible to sequentially evaluate all the others coefficients of the germ $G(z)$ from a_1. □

Remark 4.19. The Brjuno theorem shows that even stronger conclusion can be made in (2): "The number a_1 does not satisfy the Brjuno condition." The Moser theorem [72] on simultaneous linearizability also allows us to strengthen (2). Because the set of non-Diophantian multiplicators has zero measure on S^1, the set of the multiplicators of the center is infinite (it includes all numbers ϱ^n), but meager. R. Pérez-Marco [76], [77, p. 301] constructed an example of a nonlinearizable germ as in (Irr) with a center of cardinality the continuum.

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